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On the universal property of Pimsner–Toeplitz C^*-algebras and their continuous analogues

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Abstract

We consider C^*-algebras generated by a single C^*-correspondence (Pimsner–Toeplitz algebras) and by a product systems of C^*-correspondences. We give a new proof of a theorem of Pimsner, which states that any representation of the generating C^*-correspondence gives rise to a representation of the Pimsner–Toeplitz algebra. Our proof does not make use of the conditional expectation onto the subalgebra fixed under the dual action of the circle group. We then prove the analogous statement for the case of product systems, generalizing a theorem of Arveson from the case of product systems of Hilbert spaces.

Keywords: Pimsner algebras; Hilbert modules; Product systems

1. Introduction

Let E be a Hilbert module over $\mathcal{A}$, equipped with a left action of $\mathcal{A}$ via adjointable operators. We shall refer to such an E as a C^*-correspondence, or an $\mathcal{A}$-C^*-correspondence when we need to specify $\mathcal{A}$. We assume that E is full, i.e. $\overline{\langle E, E\rangle} = \mathcal{A}$. We make no further assumptions on the left action of $\mathcal{A}$. Let $\mathcal{B}$ be a C^*-algebra. A *covariant homomorphism* ψ of E into $\mathcal{B}$ is a $\mathbb{C}$-linear map $\psi_E : E \to \mathcal{B}$ along with a homomorphism $\psi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ such that for all $e, f \in E$, $a, b \in \mathcal{A}$ we have $\psi_E(aeb) = \psi_{\mathcal{A}}(a)\psi_E(e)\psi_{\mathcal{A}}(b)$, $\psi_E(e)^*\psi_E(f) = \psi_{\mathcal{A}}(\langle e, f\rangle)$. In the sequel, we will

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write ψ for both $\psi_{\mathcal{A}}$ and ψ_E, when it causes no confusion. When $\mathcal{B} = \mathcal{B}(H)$ for a Hilbert space H, we call a covariant homomorphism a *representation*.

Remark 1.1. C^*-correspondences sometimes go by the name of bimodules, and we have used the term "bimodule" for this in [H]. As term the "bimodule" is also used in the literature to indicate a situation in which there are two compatible inner products, which we do not assume exist here, we prefer to use this less ambiguous terminology here. In [MS1] what we call a *representation* is called an *isometric covariant* representation. Since we would not deal with other kinds of representations considered in [MS1], we shall use the shorter terminology.

We recall Pimsner's construction from [P]. Let $\mathcal{E} = \bigoplus_{n=0}^{\infty} E^{\otimes n}$, where by $E^{\otimes n}$ we mean the n-fold interior tensor product of E by itself over $\mathcal{A}$, and where we use the convention $E^{\otimes 0} = \mathcal{A}$ (some caution needs to be used with this convention, since while we have $E^{\otimes n} \otimes_{\mathcal{A}} E^{\otimes 0} \cong E^{\otimes n}$ for all n, we do not necessarily have $E^{\otimes 0} \otimes_{\mathcal{A}} E^{\otimes n} \cong E^{\otimes n}$ without further assumptions on the left action of $\mathcal{A}$).

For $e \in E$, let $T_e \in \mathcal{B}(\mathcal{E})$ be given by $T_e(\xi) = e \otimes \xi$. The map sending

$$e \mapsto T_e, \ \mathcal{A} \ni a \mapsto \text{left multiplication by } a$$

is a covariant homomorphism of E into $\mathcal{B}(\mathcal{E})$. We let $\mathcal{T}_E$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{E})$ generated by $\{T_e \mid e \in E\}$.

Note that if $\pi : \mathcal{T}_E \to \mathcal{B}(H)$ is a representation, then the restrictions of π to the images of E and $\mathcal{A}$ in $\mathcal{T}_E$ form a representation of E. Our goal in the first section is to give a new proof of the following theorem—a restatement of a theorem of Pimsner [P, Theorem 3.4]—which shows that any representation arises in this manner.

Theorem 1.2 (*Pimsner*). *Let ψ be a representation of E on a Hilbert space H. The map $T_e \to \psi(e)$ extends to a homomorphism $\mathcal{T}_E \to \mathcal{B}(H)$.*

Remark 1.3. Pimsner's proof relies on the conditional expectation map of the algebra $\mathcal{T}_E$ onto the fixed point subalgebra for the dual action of $\mathbb{T}$, generalizing the proof for the Cuntz algebras from [Cu]. The motivation leading to the proof presented herein was to obtain the continuous analogue, Theorem 1.10 below. The continuous analogues, described below, admit an analogous action of $\mathbb{R}$, rather than $\mathbb{T}$. Thus, one cannot obtain a bounded expectation map by averaging the group action. We note that an unbounded expectation map has been used by Zacharias to study Arveson's spectral C^*-algebras in [Z]. We refer the reader to [Ar2] for more details on the spectral C^*-algebras, and to [HZ] for a recent survey.

In [H], we considered a certain continuous analogue of the algebras $\mathcal{T}_E$, generalizing to the context of Hilbert modules Arveson's spectral C^*-algebras (see [Ar2]). We recall the definitions briefly, but refer the reader to [H] for a more technical discussion of measurable bundles of Hilbert modules.

Definition 1.4. Let $\mathcal{A}$ be a separable C^*-algebra, and $(\Omega, \mathfrak{B})$ a measurable space. A *measurable bundle of* $\mathcal{A}$ *-*C^**-correspondences over* Ω*,* E*,* is a collection $\{E_x \mid x \in \Omega\}$ of $\mathcal{A}$ -C^*-correspondences, along with a distinguished vector subspace Γ of $\Pi_{x\in\Omega} E_x$ (called the set of *measurable sections*) such that

(1) For any $\xi \in \Gamma$, $a \in \mathcal{A}$, the functions $x \mapsto \langle \xi(x), \xi(x)\rangle$, $x \mapsto \langle a\xi(x), \xi(x)\rangle$ are measurable (as functions $\Omega \mapsto \mathcal{A}$).
(2) If $\eta \in \Pi_{x\in\Omega} E_x$ satisfies that $x \mapsto \langle \xi(x), \eta(x)\rangle$ is measurable for all $\xi \in \Gamma$ then $\eta \in \Gamma$.
(3) There exists a countable subset $\xi_1, \xi_2, \ldots$ of Γ such that for all $x \in \Omega$, $\xi_1(x), \xi_2(x), \ldots$ are dense in E_x.

We denote $\mathbb{R}_+ = (0, \infty)$. Denote by E^0 the trivial $\mathbb{C}$-bundle over $\mathbb{R}_+$.

Definition 1.5. Let $\mathcal{A}$ be a separable C^*-algebra. A *product system of* $\mathcal{A}$ *-*C^**-correspondences* (or *product system of Hilbert modules over* $\mathcal{A}$) E is a measurable bundle of $\mathcal{A}$ -C^*-correspondences over $\mathbb{R}_+$, along with a multiplication map $E \times E \to E$, which descends to an isomorphism $E_x \otimes_{\mathcal{A}} E_y \to E_{x+y}$ for all $x, y \in \mathbb{R}_+$ (where E_x is the fiber over x), and is measurable in the following sense. If η, ξ are two measurable sections, we require that the section $(x, y) \mapsto \eta(x)\xi(y - x)$ (taken to mean 0 when $y \leqslant x$) be a measurable section of the bundle $E^0 \otimes E$ over $\mathbb{R}^2_+$ (whose fiber over (x, y) is $E^0_x \otimes E_y \cong \mathbb{C} \otimes E_y \cong E_y$).

Any element $e \in E_x$ gives rise to left tensoring operators $E_y \to E_{x+y}$ by $f \mapsto ef$. Those operators are adjointable, with adjoint $E_{x+y} \to E_y$ given, via the identification $E_x \otimes_{\mathcal{A}} E_y \cong E_{x+y}$, by $f \otimes g \mapsto \langle e, f\rangle g$ on elementary tensors. By abuse of notation, we denote those operators by e and e^*, for now. Each such operator gives rise, then, to a measurable family of operators on the bundle E, which in turn gives rise to adjointable operators on $\int_{\mathbb{R}_+}^{\oplus} E_x \, dx$, which we denote W_e. If ξ is a measurable section of E which is contained in $\int_{\mathbb{R}_+}^{\oplus} E_x \, dx$ (as an equivalence class), then $W_e\xi$ and $W_e^*\xi$ can also be realized as measurable sections, which we can write out explicitly: $(W_e\xi)(y) = e\xi(y - x)$ (where this is taken to be 0 if $y \leqslant x$), and $(W_e^*\xi)(y) = e^*\xi(y + x)$. Note that $\|e\|_{E_x} \geqslant \|W_e\|_{\mathcal{B}\,(\int_{\mathbb{R}_+}^{\oplus} E_x\, dx)}$. We denote by $L^1(E)$ the space of (equivalence classes of) measurable sections ξ which satisfy $\int_{\mathbb{R}_+} \|\xi(x)\| \, dx < \infty$.

Definition 1.6. For $f \in L^1(E)$ we define $W_f \in \mathcal{B}\left(\int_{\mathbb{R}_+}^{\oplus} E_x \, dx\right)$ by

$$W_f = \int_{\mathbb{R}_+} W_{f(x)} \, dx.$$

We denote by $\mathcal{W}_E$ the C^*-subalgebra of $\mathcal{B}\left(\int_{\mathbb{R}_+}^{\oplus} E_x \, dx\right)$ generated by

$$\{W_f \mid f \in L^1(E)\}.$$

We refer the reader to [H] for examples, and a discussion of the *K*-theory of $\mathcal{W}_E$.

Remark 1.7. The version of vector valued integrals we use here is the following. If X is a separable Banach space and $(\Omega, \mathfrak{B}, \mu)$ is a measure space, then a function $f : \Omega \to X$ is said to be weakly measurable if $\omega \to \varphi(f(\omega))$ is measurable for all $\varphi \in X^*$ (the dual space). Since X is separable, if f is weakly measurable, then the norm function $\omega \mapsto \|f(\omega)\|$ is measurable. Given such an f, we consider $\varphi \mapsto \int_\Omega \varphi(f(\omega))\, d\mu$. If this is a well-defined weak*-continuous linear functional on X^*, then it is given by some $x \in X$, which we define to be $\int_\Omega f(\omega)\, d\mu$. This x is guaranteed to exist if $\omega \mapsto \|f(\omega)\|$ is in $L^1(\Omega, \mu)$.

Suppose now that E is a separable Hilbert module, and $T : \Omega \to \mathcal{B}(E)$, which we write $\omega \to T_\omega$, satisfies that $\omega \to T_\omega e$ and is weakly measurable for all e. Note that this implies that $\omega \to T_\omega^* e$ is weakly measurable for all e as well (since in this case, we get the same Borel structure if we just use functionals of the form $e \mapsto \varphi(\langle f, e\rangle)$, $f \in E$, $\varphi \in \mathcal{A}^*$). Note also that it would suffice to check that $\omega \mapsto T_\omega e$ is measurable for a dense set of $e \in E$. Suppose $\omega \to \|T_\omega\|$ is in $L^1(\Omega, \mu)$, then we can define, for each $e \in E$, $\int_\Omega T_\omega e\, d\mu$ as above. This is a well-defined map $E \to E$, which we write as $\int_\Omega T_\omega\, d\mu$. This map is adjointable, with adjoint $\int_\Omega T_\omega^*\, d\mu$. When E is a Hilbert space, this coincides with the usual weak integration, which is the sense in which we shall take the integrated form of a representation in Definition 1.9 below.

We see that the operators W_f above are well defined as follows. As in the appendix of [H], we denote by Γ^2 the set of (equivalence classes of) measurable sections ξ for which $\int_{\mathbb{R}_+} \|\xi(x)\|^2\, dx < \infty$. Γ^2 is dense in $\int_{\mathbb{R}_+}^{\oplus} E_x\, dx$. We view $(x, y) \mapsto f(x)\xi(y - x)$ as section of $E^0 \otimes E$ as in the definition. So, we have that $\left\{\int_{\{x\}\times\mathbb{R}_+}^{\oplus} (E^0 \otimes E)_{(x,y)}\, dy\right\}_{x \in \mathbb{R}_+}$ is a trivial measurable bundle over $\mathbb{R}_+$, and that the section $x \mapsto \eta|_{\{x\}\times\mathbb{R}_+} \in \int_{\{x\}\times\mathbb{R}_+}^{\oplus} (E^0 \otimes E)_{x,y}\, dy$ is a measurable section of this bundle. This coincides with the section $x \mapsto T_{f(x)}\xi$, and therefore it is weakly measurable, as required.

Remark 1.8. We note that this approach is not suitable to deal with product systems over a von-Neumann algebra (unless it is finite dimensional). It seems that an appropriate modified version can be obtained for product systems over a von-Neumann algebra with a separable predual, using functionals from the predual rather than the dual space, however, we shall not deal with it in this paper.

Definition 1.9. Let E be a product system of $\mathcal{A}$-C^*-correspondences. A *representation* ψ of E on H is a map $\psi_E : E \to \mathcal{B}(H)$, along with a representation $\psi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(H)$ such that

(1) The restriction of ψ to each fiber of E is a representation of the fiber.
(2) For any $e, f \in E$, $\psi(e)\psi(f) = \psi(ef)$.
(3) If ξ is a measurable section of E then $x \mapsto \psi(\xi(x))$ is a weakly measurable function.
(4) $\overline{\bigcup_{x>0} \psi(E_x)H} \supseteq \psi(A)H$.

If $x \mapsto f(x)$ is a measurable section of E satisfying $\int_{\mathbb{R}_+} \|f(x)\|\, dx < \infty$ and ψ is a representation of E on H, then we have an integrated form of the representation

$$\psi(f) = \int_{\mathbb{R}_+} \psi(f(x))\, dx.$$

Our goal in the second part of this paper will be to prove the following continuous analogue of Theorem 1.2.

Theorem 1.10. *Let ψ be a representation of E on a Hilbert space H. The map $W_f \to \psi(f)$ extends to a homomorphism $\mathcal{W}_E \to \mathcal{B}(H)$.*

This theorem generalizes a theorem of Arveson [Ar2, Theorem 4.6.6] from the case of product systems of Hilbert spaces. Specializing our proof below to the case of Hilbert spaces will give a simpler approach to Arveson's theorem.

Remark 1.11. Arveson introduced product systems of Hilbert spaces in the context of studying semigroups of endomorphisms of $\mathcal{B}(H)$. We refer the reader to Arveson's monograph [Ar2] for further details. Alevras [Al] considered product systems of Hilbert modules, in order to study endomorphisms of type II_1 factors. There has been substantial further work on product systems of Hilbert modules in the context of endomorphism semigroups, and we refer the reader to [MS2,S], and references therein, for further details. We alert the reader to the fact that the different papers might use different definitions of a product system (the differences mainly seem to consist in the kind of measurability or continuity requirements which are imposed on the system).

2. The discrete case—proof of Theorem 1.2

Definition 2.1. Let ψ, ρ be two representations of E on H. We say that ψ *majorizes* ρ, and write $\psi \succ \rho$, if there is a (necessarily unique) homomorphism $C^*(\{\psi(e) \mid e \in E\}) \to C^*(\{\rho(e) \mid e \in E\})$ which satisfies $\psi(e) \mapsto \rho(e)$ for all $e \in E$.

In more concrete terms, $\psi \succ \rho$ means that for any $e_1, \ldots, e_n \in E$ and any polynomial p in $2n$ non-commuting variables, we have

$$\begin{aligned} &\left|\left| p(\psi(e_1), \ldots, \psi(e_n), \psi(e_1)^*, \ldots, \psi(e_n)^*)\right|\right| \\ &\quad \geqslant \left|\left| p(\rho(e_1), \ldots, \rho(e_n), \rho(e_1)^*, \ldots, \rho(e_n)^*)\right|\right|. \end{aligned}$$

We say that $T \succ \psi$ if the map $T_e \to \psi(e)$ extends to a homomorphism $\mathcal{T}_E \to C^*(\{\psi(e) \mid e \in E\})$.

Thus Theorem 1.2 states that $T \succ \psi$ for any representation ψ of E. Note that the relation $\succ$ is clearly transitive.

Definition 2.2. Let ψ be a representation of E on H. A subspace H' is said to be *invariant* for ψ if $\psi(A)H', \psi(E)H' \subseteq H'$. H' said to be *reducing* if it is invariant, and furthermore $\psi(E)^*H' \subseteq H'$.

It is always the case that $\psi(E)^*H \subseteq \psi(\mathcal{A})H$, since if $e \in E$, we can write it as $e = fa$ for some $f \in E$, $a \in \mathcal{A}$, and then for any $\xi \in H$, $\psi(e)^*\xi = \psi(a^*)\psi(f)^*\xi \in \psi(\mathcal{A})H$. Therefore $\overline{(\psi(\mathcal{A}) + \psi(E))H}$ is always reducing, and it is easy to see that the restriction of ψ to the orthocomplement of this space is the 0 representation. We shall therefore usually assume that $\overline{(\psi(\mathcal{A}) + \psi(E))H} = H$. A representation satisfying this condition will be called *non-degenerate*. If the left action of $\mathcal{A}$ on E is non-degenerate (i.e. $\mathcal{A}E = E$) then we also have $\psi(E)H \subseteq \psi(\mathcal{A})H$, in which case this non-degeneracy condition is equivalent to just saying that $\psi(\mathcal{A})H = H$.

We first recall the following lemma (noted in [MS1] and in references therein). The proof is straightforward.

Lemma 2.3. *Let ψ be a representation of E on H. Regarding H as a left $\mathcal{A}$-module via ψ, we form the tensor product $E \otimes_{\mathcal{A}} H$ to obtain a Hilbert space. The contraction map*

$$e \otimes \xi \mapsto \psi(e)\xi \quad e \in E, \; \xi \in H$$

extends to an isometry

$$E \otimes_{\mathcal{A}} H \to H$$

If ψ is a representation of E, $n > 0$, then we can define a representation of $E^{\otimes n}$ by $e_1 \otimes \cdots \otimes e_n \mapsto \psi(e_1)\psi(e_2)\cdots\psi(e_n)$. We will denote this representation by ψ as well.

If π is a representation of $\mathcal{A}$ on a Hilbert space H, then we can define a representation $T \otimes_{\mathcal{A}} 1$ of E on $\mathcal{E} \otimes_\pi H$ (this is called an *induced* representation in [MS1]). We know from the general theory of Hilbert modules that we have natural homomorphism $\mathcal{B}(\mathcal{E}) \to B(\mathcal{E} \otimes_\pi H)$ given by $A \mapsto A \otimes 1_H$. This immediately implies that $T \succ T \otimes_{\mathcal{A}} 1$.

The following lemma (see [MS1]) generalizes the part of the Wold decomposition which says that any isometry S which satisfies $S^n S^{n*} \to 0$ in the strong operator topology is unitarily equivalent to a direct sum of copies of the unilateral shift on ℓ^2. We state it here in a form appropriate for our use.

Lemma 2.4. *Let ψ be a non-degenerate representation of E on H, such that $\bigcap_{n>0} \overline{\psi(E^{\otimes n})H} = \{0\}$, and let $H_0 = (\psi(E)H)^\perp$.*

(1) *H_0 is invariant for $\psi(\mathcal{A})$.*

(2) *Let $H_n = \overline{\psi(E^{\otimes n})H_0}$. We have $H = \bigoplus_{n=0}^{\infty} H_n$.*

(3) *For any n, $C_n : E^{\otimes n} \otimes_{\mathcal{A}} H_0 \to H_n$ given by the contraction*

$$C_n(e_1 \otimes \cdots \otimes e_n \otimes \xi) = \psi(e_1)\psi(e_2)\cdots\psi(e_n)\xi$$

is a well-defined unitary operator (*where for* C_0 *is the contraction* $a \otimes \xi \mapsto \psi(a)\xi$, $a \in E^{\otimes 0} = \mathcal{A}$).

(4) $C = \bigoplus_{n=0}^{\infty} C_n : \mathcal{E} \otimes_{\mathcal{A}} H_0 \to H$ *is a unitary operator which satisfies*

$$C(T_e \otimes_{\mathcal{A}} 1) = \psi(e)C$$

for all $e \in E$, *i.e. it implements a unitary equivalence between the representations* $T \otimes_{\mathcal{A}} 1_{H_0}$ *and* ψ.

Corollary 2.5. *Let* ψ *be as in Lemma* 2.4, *then* $T \succ \psi$.

Now let ψ be any representation of E on H. By Corollary 2.5, to prove Theorem 1.2, it suffices to show that ψ is majorized by a representation which satisfies the condition of Lemma 2.4.

For any $\lambda \in \mathbb{T}$, we define a representation ψ_λ, given by $\psi_\lambda(e) = \lambda\psi(e)$, $\psi_\lambda(a) = \psi(a)$, $e \in E$, $a \in \mathcal{A}$. We can now form a direct integral to obtain a representation $\int_{\mathbb{T}}^{\oplus} \psi_\lambda d\lambda$ on $H \otimes L^2(\mathbb{T})$. In those terms, it is given by $a \mapsto \psi(a) \otimes 1$, $e \mapsto \psi(e) \otimes M_z$ (where M_z is the multiplication operator by the inclusion function $\mathbb{T} \to \mathbb{C}$). Since $\psi_\lambda(e) \to \psi(e)$ as $\lambda \to 1$ for all e (in norm), we can easily see that $\int_{\mathbb{T}}^{\oplus} \psi_\lambda d\lambda \succ \psi$.

Let U be the bilateral shift on $\ell^2(\mathbb{Z})$. We form a representation $\psi \otimes U$ of E on $H \otimes \ell^2(\mathbb{Z})$ by $(\psi \otimes U)(e) = \psi(e) \otimes U$, $(\psi \otimes U)(a) = \psi(a) \otimes 1$, $e \in E$, $a \in \mathcal{A}$. Those two representations are unitarily equivalent (by applying the Fourier transform to the second variable), so we have $\psi \otimes U \succ \psi$.

Denote by P_+ the projection of $\ell^2(\mathbb{Z})$ onto $\ell^2(\mathbb{N})$. Denote by S the unilateral shift on $\ell^2(\mathbb{N})$ (which we also think of as $S = UP_+$ on $\ell^2(\mathbb{Z})$), and let $V = 1_H \otimes S$.

Denote by $\psi \otimes S$ the restriction of $\psi \otimes U$ to the invariant subspace $H \otimes \ell^2(\mathbb{N})$, i.e. $(\psi \otimes S)(\cdot) = (\psi \otimes U)(\cdot)(1 \otimes P_+)$ (where here we will think of $\psi \otimes S$ as both a representation on $H \otimes \ell^2(\mathbb{N})$ and as a (degenerate) representation on $H \otimes \ell^2(\mathbb{Z})$).

Observation 2.6. *For any* k, *any polynomial* $p(x_1, \ldots, x_k, y_1, \ldots, y_k)$ *in* $2k$ *non-commuting variables and any* $e_1, \ldots, e_k \in E$, *and any* $m > \deg(p)$, *we have*

$$\begin{aligned}(1 \otimes S)^{m*} & p((\psi \otimes S)(e_1), \ldots, (\psi \otimes S)(e_k)^*)(1 \otimes S)^m \\ &= (1 \otimes P_+)p((\psi \otimes U)(e_1), \ldots, (\psi \otimes U)(e_k)^*)(1 \otimes P_+)\end{aligned}$$

(*where those are thought of as operators on* $H \otimes \ell^2(\mathbb{Z})$).

Proof. The proof is elementary, and involves just manipulating the second variable. It suffices to prove this for $p(x_1, \ldots, x_k, y_1, \ldots, y_k)$ a non-commutative monomial. We list the arguments so that each x variable corresponds to an element of E under the relevant representation, and each y variable corresponds to the adjoints.

Let $\xi \in H$, $n \in \mathbb{Z}$. We check that both sides agree when applied to $\xi \otimes \delta_n$, δ_n being the sequence in $\ell^2(\mathbb{Z})$ which is 1 at n and 0 otherwise. Both sides annihilate $\xi \otimes \delta_n$ for $n < 0$, so it suffices to consider $n \geqslant 0$.

Note that

$$(\psi \otimes U)(e)(\xi \otimes \delta_n) = \psi(e)\xi \otimes \delta_{n+1}, (\psi \otimes S)(e)(\xi \otimes \delta_n) = \begin{cases} \psi(e)\xi \otimes \delta_{n+1} & | \quad n \geqslant 0, \\ 0 & | \ n < 0, \end{cases}$$

$$(\psi \otimes U)(e)^*(\xi \otimes \delta_n) = \psi(e)^*\xi \otimes \delta_{n-1},$$

$$(\psi \otimes S)(e)^*(\xi \otimes \delta_n) = \begin{cases} \psi(e)^*\xi \otimes \delta_{n-1} & | \ n > 0, \\ 0 & | \quad n \leqslant 0, \end{cases}$$

so for $n > 0$ the expressions involving S agree with those involving U. Let $A = p(\psi(e_1), \ldots, \psi(e_k)^*)$, $A_S = p((\psi \otimes S)(e_1), \ldots, (\psi \otimes S)(e_k)^*)$, $A_U = p((\psi \otimes U)(e_1), \ldots, (\psi \otimes U)(e_k)^*)$ and let j be the difference between the sum of the exponents of the x variables and the sum of the exponents of the y variables. So if l is greater than the sum of the exponents of the y variable in our monomial, then we have $A_S(\xi \otimes \delta_l) = A_U(\xi \otimes \delta_l) = (A\xi) \otimes \delta_{l+j}$. This will be satisfied automatically if $l \geqslant m$. So, the right-hand side applied to $\xi \otimes \delta_n$ is $(1 \otimes P_+)(A\xi \otimes \delta_{n+j}) = \begin{cases} A\xi \otimes \delta_{n+j} & | \quad n + j \geqslant 0, \\ 0 & | \ n + j < 0. \end{cases}$ The left-hand side is $(1 \otimes S)^{m*} A_S(\xi \otimes \delta_{n+m})$, and since $n + m \geqslant m$, this in turn equals $(1 \otimes S)^{m*} A\xi \otimes \delta_(n + m + j) = \begin{cases} A\xi \otimes \delta_{n+j} & | \quad n + j \geqslant 0, \\ 0 & | \ n + j < 0, \end{cases}$ as required. $\square$

Lemma 2.7. *If A is in the $*$-algebra generated by $\{(\psi \otimes U)(e) \mid e \in E\}$ then $\|(1 \otimes P_+)A(1 \otimes P_+)\| = \|A\|$.*

Proof. Note that any operator of the form $(\psi \otimes U)(e)$ commutes with all operators of the form $1_H \otimes U^n$, $n \in \mathbb{Z}$. Therefore A commutes with $1 \otimes U^n$, $n \in \mathbb{Z}$ as well. Let P_n denote the projection onto $H \otimes \ell^2(\{n, n+1, \ldots\})$ (so $P_0 = 1 \otimes P_+$). We have $(1 \otimes U^m)P_n = P_{n+m}(1 \otimes U^m)^*$ for all $n, m \in \mathbb{Z}$, and therefore we have

$$(1 \otimes U^n)P_0 A P_0 (1 \otimes U^n)^* = P_n (1 \otimes U^n) A (1 \otimes U^n)^* P_n = P_n A P_n,$$

so $\|P_0 A P_0\| = \|P_n A P_n\|$ for all $n \in \mathbb{Z}$. Since $P_n \to 1_{\ell^2(\mathbb{Z})}$ as $n \to -\infty$ in the strong operator topology, we have

$$\|P_0 A P_0\| = \lim_{n \to -\infty} \|P_n A P_n\| = \|A\|$$

as required. $\square$

Corollary 2.8. $\psi \otimes S \succ \psi \otimes U$.

Proof. Let $e_1, \ldots, e_k \in E$, and let $p(x_1, \ldots, x_{2k})$ be a polynomial in $2k$ non-commuting variables. Since the $(1 \otimes S)^m$ are isometries, we have

$$\|(1 \otimes S)^{m*} p((\psi \otimes S)(e_1), \ldots, (\psi \otimes S)(e_k)^*)(1 \otimes S)^m\|$$
$$\leqslant \|p((\psi \otimes S)(e_1), \ldots, (\psi \otimes S)(e_k)^*)\|$$

so by Observation 2.6, we have

$$\|(1 \otimes P_+) p((\psi \otimes U)(e_1), \ldots, (\psi \otimes U)(e_k)^*)(1 \otimes P_+)\|$$
$$\leqslant \|p((\psi \otimes S)(e_1), \ldots, (\psi \otimes S)(e_k)^*)\|$$

and by Lemma 2.7,

$$\|(1 \otimes P_+) p((\psi \otimes U)(e_1), \ldots, (\psi \otimes U)(e_k)^*)(1 \otimes P_+)\|$$
$$= \|p((\psi \otimes U)(e_1), \ldots, (\psi \otimes U)(e_k)^*)\| \qquad \square$$

Proof of Theorem 1.2. Note that $\psi \otimes S$ satisfies the conditions of Lemma 2.4. It therefore suffices to show that $\psi \otimes S \succ \psi$, and indeed, we saw that $\psi \otimes S \succ \psi \otimes U$ and $\psi \otimes U \succ \psi$. $\square$

Remark 2.9. The proof in this section was obtained in the course of the author's dissertation work under the supervision of W.B. Arveson, and is motivated by ideas from [Ar2].

3. The continuous case—proof of Theorem 1.10

The approach here will differ from the proof above for the discrete case, in that we do not have a continuous analogue of Lemma 2.4 (see Remark 3.9 below). Aside for that, we shall follow a similar path.

We begin by giving the analogue of Definition 2.1.

Definition 3.1. Let ψ, ρ be two representations of a product system E (over $\mathcal{A}$) on H. We say that ψ *majorizes* ρ, and write $\psi \succ \rho$, there is a (necessarily unique) homomorphism $C^*(\{\psi(f) \mid f \in L^1(E)\}) \to C^*(\{\rho(f) \mid f \in L^1(E)\})$ which satisfies $\psi(f) \mapsto \rho(f)$ for all $f \in L^1(E)$. If $\psi \succ \rho$ and $\rho \succ \psi$, we write $\psi \approx \rho$.

We say that $W \succ \psi$ if the map $W_f \to \psi(f)$ extends to a homomorphism $\mathcal{W}_E \to C^*(\{\psi(f) \mid f \in L^1(E)\})$.

The relation $\succ$ is clearly transitive. Theorem 1.10 states that $W \succ \psi$ for any representation ψ of E.

Definition 3.2. Let ψ be a representation of E on H. A subspace H' of H is said to be *invariant* for ψ if $\psi(\mathcal{A})H' \subseteq \overline{\bigcup_{x>0} \psi(E_x)H'} \subseteq H'$. H' will be said to be *reducing* if it is invariant, and furthermore $\psi(E_x)^* H' \subseteq H'$ for all $x > 0$.

Let ψ be a representation of E on H, and let H' be invariant for ψ, then we have a representation of E on H' by restriction. Let P be the projection onto H', and let ψ' denote the restriction, then $\psi'(f) = \psi(f)P$. Notice that if H' is furthermore reducing, then $\psi \succ \psi'$.

Suppose $A_1, \ldots, A_n,\ A_1^{(j)}, \ldots, A_n^{(j)},\ j = 1, 2, \ldots$ are operators in $B(H)$ such that $A_k^{(j)} \to A_k$ in the strong operator topology, $k = 1, \ldots, n$. If $p(x_1, \ldots, x_n)$ is a non-commutative polynomial, then $p\left(A_1^{(j)}, \ldots, A_n^{(j)}\right) \to p(A_1, \ldots, A_n)$ in the strong operator topology. Thus, if $\left\|p\left(A_1^{(j)}, \ldots, A_n^{(j)}\right)\right\| \leqslant M$ for all j, then $\|p(A_1, \ldots, A_n)\| \leqslant M$. We therefore have the following approximation lemma.

Lemma 3.3. *Let ψ be a representation of E on H. Suppose that there is a sequence of projections $P_n \to 1$ in the strong operator topology, such that $P_n H$ is invariant for ψ for all n. Denote by ψ_n the restricted representation of ψ to $P_n H$. For any polynomial $p(x_1, \ldots, x_{2k})$ in $2k$ non-commuting variables and $f_1, \ldots, f_k \in L^1(E)$, if $\left\|p(\psi_n(f_1), \ldots, \psi_n(f_k), \psi_n(f_1)^*, \ldots, \psi_n(f_k)^*)\right\| \leqslant M$ for all n then $\left\|p(\psi(f_1), \ldots, \psi(f_k), \psi(f_1)^*, \ldots, \psi(f_k)^*)\right\| \leqslant M$.*

Consequently, if ρ is a representation of E such that $\rho \succ \psi_n$ for all n then $\rho \succ \psi$.

If π is a representation of $\mathcal{A}$ on H, we can form a representation $W \otimes_{\mathcal{A}} 1$ of $\mathcal{W}_E$ on $\int_{\mathbb{R}_+}^{\oplus} E_x\, dx \otimes_\pi H$, as in the discrete case, and the same argument shows that $W \succ W \otimes_{\mathcal{A}} 1$.

Let ψ be a representation of E on H. Denote $H_x = \overline{\psi(E_x)H}$. We have $E_x \otimes_{\psi_{\mathcal{A}}} H \cong H_x$ via the contraction map $e \otimes \xi \mapsto \psi(e)\xi$.

Note that $\psi(E_x)H_y \subseteq H_{x+y}$, $\psi(E_x)^* H_{x+y} \subseteq H_y$, and that if $x > y > 0$ then $H_x \subseteq H_y$. We always have $\psi(E_x)^* H \subseteq \psi(A)H$. Since $\overline{\bigcup_{x>0} H_x} \supseteq \psi(\mathcal{A})H$ (by the last requirement in Definition 1.9), we see, then, that $\overline{\bigcup_{x>0} H_x}$ is reducing, and that the restriction of ψ to the orthocomplement of this subspace is 0. We call ψ *non-degenerate* if $\overline{\bigcup_{x>0} H_x} = H$, and note that to prove our theorem, we can assume that ψ is non-degenerate.

We may identify the Hilbert spaces $\int_{\mathbb{R}_+}^{\oplus} E_x\, dx \otimes_{\mathcal{A}} H$ with $\int_{\mathbb{R}_+}^{\oplus} E_x \otimes_{\mathcal{A}} H\, dx$, as follows. We check that the map which sends $f \otimes_{\mathcal{A}} \xi$, where $f \in L^1(E)$ satisfies $\int_{\mathbb{R}_+} \|f(x)\|^2\, dx < \infty$, to the section $x \mapsto f(x) \otimes_{\mathcal{A}} \xi$ (of the measurable bundle whose fiber over x is $E_x \otimes_{\mathcal{A}} H$) extends to a unitary operator, and we identify the two spaces by means of this unitary. Note that the bundle of Hilbert spaces whose fiber over x is H_x is a measurable sub-bundle of the trivial bundle $\mathbb{R}_+ \times H$, and further applying the contraction map to each fiber, we obtain a unitary

$$C : \int_{\mathbb{R}_+}^{\oplus} E_x\, dx \otimes_{\mathcal{A}} H \to \int_{\mathbb{R}_+}^{\oplus} H_x\, dx \subseteq H \otimes L^2(\mathbb{R}_+) \cong L^2(\mathbb{R}_+, H).$$

Let $S_x : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ denote the unilateral shift semigroup. We form a representation $\psi \otimes S$ on $H \otimes L^2(\mathbb{R}_+)$ by $(\psi \otimes S)(e) = \psi(e) \otimes S_x$ $(e \in E_x)$, $(\psi \otimes S)(a) = \psi(a) \otimes 1$.

Let $H_\# = \int_{\mathbb{R}_+}^{\oplus} H_x\, dx = \{\xi \in L^2(\mathbb{R}_+, H) \mid \xi(x) \in H_x \text{ a.e. } x\}$. Using the behaviour of the spaces H_x discussed above, it is easy to check that $H_\#$ is reducing for $\psi \otimes S$. We denote the restriction of $\psi \otimes S$ to $H_\#$ by $\psi_\#$.

Lemma 3.4. *Let ψ be a non-degenerate representation of E on a Hilbert space H. Using the above notation, the operator C implements a unitary equivalence between the representations $\psi_\#$ on $H_\#$ and $W \otimes_{\mathcal{A}} 1$ of E on $\int_{\mathbb{R}_+}^{\oplus} E_x\, dx \otimes_{\mathcal{A}} H$.*

Proof. We need to check that $C(W_e \otimes_{\mathcal{A}} 1) = (\psi(e) \otimes S_x)C$ for all $e \in E_x$, $x \in \mathbb{R}_+$, and that $C(W_a \otimes_{\mathcal{A}} 1) = (\psi(a) \otimes 1)C$ for all $a \in \mathcal{A}$. We show this for $e \in E_x$ – the case of $a \in \mathcal{A}$ is similar. It suffices to check this for vectors of the form $f \otimes_{\mathcal{A}} \xi$ where $f \in L^1(E)$ satisfies $\int_{\mathbb{R}_+} \|f(x)\|^2\, dx < \infty$ as above (since those are total). Indeed, $C(W_e \otimes_{\mathcal{A}} 1)(f \otimes_{\mathcal{A}} \xi)(y) = C(ef \otimes_{\mathcal{A}} \xi)(y) = \psi((e \cdot f)(y))\xi = \psi(e \cdot f(y-x))\xi = \psi(e)\psi(f(y-x))\xi = (\psi(e) \otimes S_x)(C(f \otimes_{\mathcal{A}} \xi))$, as required (where $f(y-x)$ is understood to mean 0 if $x > y$). $\square$

Lemma 3.5. *Let ψ be a non-degenerate representation of E on H. Let $\psi \otimes S$, $\psi_\#$, $H_\#$ be as in Lemma 3.4, then $\psi \otimes S \approx \psi_\#$.*

Proof. Since $H_\#$ is a reducing subspace, we have $\psi \otimes S \succ \psi_\#$. Thus it remains to show that $\psi_\# \succ \psi \otimes S$. By Lemma 3.3, it suffices to exhibit projections $P_\varepsilon \in \mathcal{B}(L^2(\mathbb{R}_+, H))$ such that $P_\varepsilon L^2(\mathbb{R}_+, H)$ is invariant for $\psi \otimes S$, $P_\varepsilon \to 1$ as $\varepsilon \to 0$ (in the strong operator topology), and $\psi_\# \succ \psi_\varepsilon$ where ψ_ε denotes the restriction of $\psi \otimes S$ to $P_\varepsilon(L^2(\mathbb{R}_+, H)$.

Let $K_\varepsilon = \{\xi \in L^2(\mathbb{R}_+, H) \mid \xi(x) \in H_x \ominus H_{x+\varepsilon}\} \subseteq H_\#$. K_ε is reducing for $\psi_\#$ (and for $\psi \otimes S$). Denote the restriction of $\psi_\#$ to K_ε by ψ_ε^0.

Now, for $n = 1, 2, \ldots$, let $K_\varepsilon^n = \{\xi \in L^2(\mathbb{R}_+, H) \mid \xi(x) \in H_{x-n\varepsilon} \ominus H_{x-(n-1)\varepsilon}\}$, where we take H_x to be the 0 space if $x < 0$ (so the sections $\xi \in K_\varepsilon^n$ are required to be 0 a.e. on the interval $(0, n\varepsilon)$). Note that the K_ε^n are mutually orthogonal, and are all orthogonal to $H_\#$. K_ε^n is invariant (but not reducing) for $\psi \otimes S$. Let ψ_ε^n denote the restriction of $\psi \otimes S$ to K_ε^n.

Define $U_\varepsilon^n : K_\varepsilon \to K_\varepsilon^n$ by $U_\varepsilon^n(\xi)(x) = \xi(x - n\varepsilon)$ (where $\xi(x - n\varepsilon)$ is understood to be 0 if $x \leqslant n\varepsilon$). It is easy to check that U_ε^n is unitary, and implements a unitary equivalence between ψ_ε^0 and ψ_ε^n. In particular, we have $\psi_\# \succ \psi_\varepsilon^n$ for all n.

Let $H^\varepsilon = H_\# \oplus \bigoplus_{n=1}^{\infty} K_\varepsilon^n \subseteq L^2(\mathbb{R}_+, H)$. Since it is a direct sum of invariant subspaces, H^ε is invariant for $\psi \otimes S$ as well. Let P_ε be the projection onto the H^ε, and ψ_ε the restriction of $\psi \otimes S$ to H^ε. So, $\psi_\varepsilon = \psi_\# \oplus \bigoplus_{n=1}^{\infty} \psi_\varepsilon^n$, and we have $\psi_\# \succ \psi_\varepsilon$.

So $P_\varepsilon \to 1$ (since, for example, the range of P_ε contains $L^2(\mathbb{R}_+, H_\varepsilon)$, and we know that H_ε increase to H as $\varepsilon \to 0$ by the non-degeneracy assumption), and $\psi_\# \succ \psi_\varepsilon$ for all ε, which is what we needed. $\square$

We may now proceed as in the discrete case. For $\lambda \in \mathbb{R}$, we define a representation ψ_λ by $\psi_\lambda(e) = e^{ix\lambda}\psi(e)$, $\psi_\lambda(a) = \psi(a)$. We form a representation $\int_{\mathbb{R}}^{\oplus} \psi_\lambda \, d\lambda$ $H \otimes L^2(\mathbb{R})$, and since $\psi_\lambda(e) \to \psi(e)$ as $\lambda \to 0$ for all e (in norm), we have $\int_{\mathbb{R}}^{\oplus} \psi_\lambda \, d\lambda \succ \psi$.

Let U_x be the bilateral shift group on $L^2(\mathbb{R})$. We form a representation $\psi \otimes U$ of E on $H \otimes L^2(\mathbb{R})$ by $(\psi \otimes U)(e) = \psi(e) \otimes U_x$ $(e \in E_x)$, $(\psi \otimes U)(a) = \psi(a) \otimes 1$. Using the Fourier transform, we see that $\int_{\mathbb{R}}^{\oplus} \psi_\lambda \, d\lambda$ and $\psi \otimes U$ are unitarily equivalent, so $\psi \otimes U \succ \psi$.

Let P_0 the projection of $H \otimes L^2(\mathbb{R})$ onto $H \otimes L^2(\mathbb{R}_+)$

The following are immediate analogues of Observation 2.6 and Lemma 2.7 above (and immediate generalizations of 4.5.3 and 4.5.4 in [Ar2]). We leave the simple proofs to the reader.

Observation 3.6. *For any k and any polynomial $p(x_1, \ldots, x_{2k})$ in $2k$ non-commuting variables and any $f_1, \ldots, f_k \in L^1(E)$, we have*

$$\lim_{x \to \infty} \Big\| (1 \otimes S_x)^* p((\psi \otimes S)(f_1), \ldots, (\psi \otimes S)(f_k)^*)(1 \otimes S_x) - P_0 \, p((\psi \otimes U)(f_1), \ldots, (\psi \otimes U)(f_k)^*) \, P_0 \Big\| = 0$$

Lemma 3.7. *If A is in the $*$-algebra generated by $\{(\psi \otimes U)(f) \mid f \in L^1(E)\}$ then $\|P_0 A P_0\| = \|A\|$.*

Corollary 3.8. $\psi \otimes S \succ \psi \otimes U$.

Proof of Theorem 1.10. Let ψ be a (non-degenerate) representation of E on H. We want to show that $W \succ \psi$. So, $W \succ W \otimes_{\mathcal{A}} 1$, $W \otimes_{\mathcal{A}} 1 \approx \psi_\#$ (by Lemma 3.4), $\psi_\# \approx \psi \otimes S$ (by Lemma 3.5), $\psi \otimes S \succ \psi \otimes U$ (Corollary 3.8), and $\psi \otimes U \succ \psi$ (as remarked above), concluding the argument. $\square$

Remark 3.9. There is a continuous analogue of the Wold decomposition, due to Cooper [Co], which states that if S_x $(x > 0)$ is a strongly continuous semigroup of isometries on a Hilbert space, then S_x is unitarily equivalent to a direct sum of a one-parameter unitary group and copies of the unilateral shift semigroup on $L^2(\mathbb{R}_+)$.

Unlike the case of a single C^*-correspondence, this does not quite generalize to product systems. There is an approximate version, due to Arveson [Ar1,Ar2] for product systems of Hilbert spaces. Arveson's theorem states the following. Let E is a product system of Hilbert *spaces*, and let ψ be a representation of E on H such that $\bigcap_{x>0} \overline{\psi(E_x)H} = \{0\}$. For any $\varepsilon > 0$, let $H_\varepsilon = \overline{\psi(E_\varepsilon)H}$, and let ψ_ε be the restriction of ψ to H_ε, then ψ_ε is unitarily equivalent to a direct sum of copies of the representation W of E on $\int_{\mathbb{R}_+}^{\oplus} E_x \, dx$ ($\int_{\mathbb{R}_+}^{\oplus} E_x \, dx$ here would be a Hilbert space, so W is a representation, called the *regular* representation). However, Arveson showed in [Ar1] that H_ε cannot be replaced by H in the theorem. Arveson's proof of the special case of Theorem 1.10 for Hilbert spaces makes use of this Cooper-type theorem, and is more complicated than the proof given in this paper. We do not know if the natural generalization of Arveson's theorem to the case of Hilbert modules holds.

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